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ALPHA POWER TRANSFORMED LOMAX-EXPONENTIAL DISTRIBUTION: PROPERTIES AND APPLICATION TO BLADDER CANCER DATA.

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Abstract

The Lomax-Exponential distribution has been shown to provide improved flexibility over the classical exponential distribution for modelling lifetime data; however, additional flexibility is often required to adequately capture complex survival patterns encountered in practice. This study proposes a new lifetime distribution called the Alpha Power Transformed Lomax-Exponential Distribution (APTLED) by applying the alpha power transformation to the Lomax-Exponential distribution. The proposed model introduces an additional shape parameter that enhances its ability to model diverse distributional shapes and hazard rate behaviours. Fundamental statistical properties of the distribution, including the probability density function, cumulative distribution function, moments, moment generating function, characteristic function, survival function and hazard rate function, are derived. A simplified form of the probability density function is also developed to facilitate theoretical derivations. The unknown model parameters are estimated using the maximum likelihood estimation method. The practical usefulness of the proposed distribution is demonstrated using remission times of bladder cancer patients and compared with several existing extensions of the exponential and inverse exponential distributions. The performance of the proposed model is evaluated Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Anderson-Darling, Cramér-von Mises and Kolmogorov-Smirnov goodness-of-fit statistics. Results indicate that the APTLED provides the best overall fit among all competing models, demonstrating its superior flexibility and suitability for modelling positively skewed lifetime data encountered in reliability and survival analysis.

Keywords: Alpha Power Transformation, Lomax-Exponential Distribution, Lifetime Distribution, Maximum Likelihood Estimation, Survival Analysis, Hazard Rate Function, Bladder Cancer Data.

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1 Introduction

Power transformations are a cornerstone of modern statistical modeling because they can reduce skewness, stabilize variance, and increase model flexibility when fitting real-life data (Box & Cox, 1964; Yeo & Johnson, 2000). The alpha power transformation method introduced by Mahdavi and Kundu (2017) as a general method for generating new univariate distributions has recently become a widely used formula for deriving highly flexible lifetime distributions, producing models with richer shapes for densities and hazard rates than many classical alternatives (Mahdavi & Kundu, 2017; Nassar et al., 2017).

Researchers have applied alpha power transformed family across many base distributions and it has been showing systematic improvements in fit and interpretability for applied problems in reliability, survival analysis, and other empirical domains (Ahmad et al., 2020; Elbatal et al., 2022; Ihtisham et al., 2019; Ali et al., 2021; Ahmad et al., 2021; Ceren et al., 2018; Eghwerido, 2021; Mohiuddin and Kannan, 2021; Dey et al., 2019; Agegnehu et al., 2024; Dugasa, 2024).

The exponential distribution is used in Poisson processes and gives a description of the time between events. It is applied mostly in life testing experiments. However, it has memoryless property with a constant failure rate which makes the distribution unsuitable for modelling real-life problems and hence creating a vital problem in statistical modeling and applications (Lemonte, 2013). To make up for these limitations, Keller and Kamath (1982) proposed a modified version of the exponential distribution, called the inverse exponential distribution which has also been studied in details by Lin *et al.*, (1989). The inverse exponential distribution was found adequate for modeling datasets with inverted bathtub failure rates (Keller and Kamath, 1982) but this modified version also has a limitation which is its inability to efficiently capture datasets that are highly skewed (Abouammoh and Alshingiti, 2009). The cumulative distribution function (cdf) and probability density function (pdf) of the exponential distribution are respectively defined as:

$$G(x) = 1 - e^{-\theta x} \tag{1}$$

and

$$g(x) = \theta e^{-\theta x} \tag{2}$$

where $x, \theta > 0$, and θ is the scale parameter of the exponential distribution while X is the random variable.

To address these limitations, researchers have developed many families of distributions to enhance flexibility by incorporating additional shape parameters while retaining analytical tractability. Notable among these families include the quadratic rank transmutation map by Shaw and Buckley (2007), the exponentiated T-X family by Alzaghal *et al*. (2013), the Weibull-X family by Alzaatreh *et al*. (2013), the Weibull-G family by Bourguignon *et al*. (2014), the Lindley-G family by Cakmakyapan and Ozel (2016), the odd Lindley-G family by Gomes-Silva *et al*. (2017), the odd Lomax-G family by Cordeiro *et al.* (2019), the odd Chen-G family of distributions by Anzagra *et al*. (2022), a novel bivariate Lomax-G family of distributions Fayomi *et al*. (2023), the X-exponential-G family of distributions by Mohammad (2024) and the new Frechet-G family by Ieren et al. (2024).

These families of distributions have been used by many researchers to develop flexible compound probability distributions including extended forms of the exponential and inverse exponential distributions such as the Weibull-exponential distribution by Oguntunde *et al*, (2015), the transmuted Weibull-exponential distribution by Yahaya and Ieren (2017), the exponential inverse exponential distribution by Oguntunde et al. (2017), the transmuted exponential-inverse exponential distribution by Ieren et al. (2020), transmuted Lindley-exponential distribution by Umar *et al*., (2019), odd Lindley-inverse exponential distribution by Ieren and Abdullahi (2020), the Lomax-exponential distribution by Ieren and Kuhe (2018), the transmuted odd generalized exponential-exponential distribution by Abdullahi et al. (2018), the transmuted exponential distribution by Owoloko *et al*. (2015), the odd Lomax-inverse exponential distribution by

Ieren et al. (2021), the Lomax- inverse exponential distribution by Abdulkadir *et al*. (2020), the exponentiated exponential-inverse exponential distribution by Asongo et al. (2020), the Weibull-exponential inverse exponential distribution by Chama et al. (2021)and sine Lomax-exponential distribution by Joel et al. (2024). These extended distributions have been found to be better than the conventional or classical ones.

The probability density function (pdf) of the Lomax-Exponential distribution (LED) according to Ieren and Kuhe (2018) is defined by

$$g(x) = ab^a\theta\left(b+\theta x\right)^{-(a+1)} \tag{3}$$

The corresponding cumulative distribution function (CDF) of LED is given by

$$G(x) = 1 - b^a\left(b+\theta x\right)^{-a} \tag{4}$$

where, $x>0, \theta>0, a>0, b>0$; a and b are shape parameters and θ is a scale parameter. According to Ieren and Kuhe (2018), the distribution is found to be better than the Weibull-exponential distribution and exponential distribution after considering three applications to real life data.

This paper develops and studies the Alpha Power Transformed Lomax-exponential distribution (APTLED). The remaining sections of this paper are as follows: definition of the new distribution with its plots is provided in section 2. Section 3 presents the simplification of the pdf of the APTLED. Section 4 derived some properties of the proposed distribution. The estimation of parameters using maximum likelihood estimation (MLE) is presented in section 5. An application of the new model with other existing distributions to a real-life bladder cancer data is captured in section 6 and the conclusion is given in section 7.

2. Alpha Power Transformed Lomax-Exponential distribution (APTLED)

The cumulative distribution function (cdf) and the probability density function (pdf) of the Alpha Power transformed family of distributions by Mahdavi and Kundu (2017), are defined as:

$$F(x) = \frac{\alpha^{G(x)}-1}{\alpha-1} \tag{5}$$

and

$$f(x) = \frac{\log(\alpha)}{(\alpha-1)} g(x)\alpha^{G(x)} \tag{6}$$

"respectively, where $g(x)$ and $G(x)$ are the *pdf* and the *cdf* of any continuous distribution to be modified respectively and $\alpha>0$ $(\alpha\neq 1)$ is the power or shape parameter of the family responsible for additional skewness and flexibility in the modified model.

Substituting equations (3) and (4) into equations (5) and (6) and simplifying, we obtain the cdf and pdf of the APTLED (for $x>0$) are given in equations (7) and (8) respectively as follows:

$$F(x) = (\alpha-1)^{-1}\left(\alpha^{\left(1-b^a(b+\theta x)^{-a}\right)}-1\right) \tag{7}$$

and

$$f(x)=\frac{\log(\alpha)}{(\alpha-1)}ab^{a}\theta(b+\theta x)^{-(a+1)}\alpha^{\left(1-b^{a}(b+\theta x)^{-a}\right)} \quad (8)$$

where $a>0$ and $b>0$ are the scale and shape parameters of the APTLED respectively, and $\alpha>0$ is the alpha power transformed parameter.

Figure 1 below presents the plots of the pdf and cdf of the APTLED using some parameter values as follows.

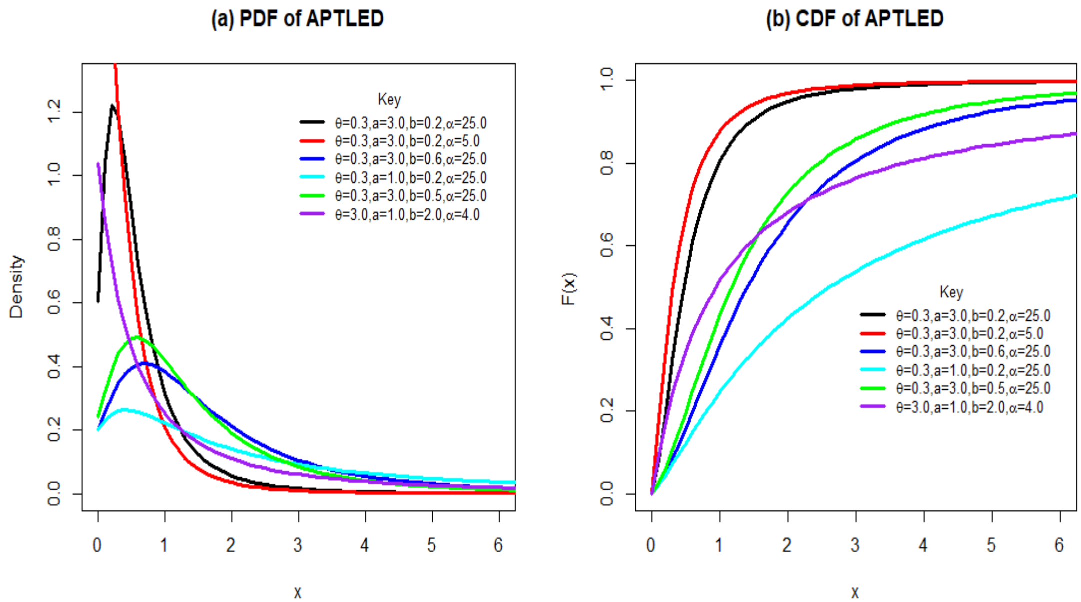

Figure 1: PDF and CDF of the APTLED.

The plot of the pdf in figure 1(a) above revealed that the pdf APTLED is positively skewed and flexible with different shapes depending on the chosen values of the parameters. Also, the plot of the cdf in figure 1(b) indicates that it is a valid distribution because the value of the cdf equals one when X approaches infinity and equals zero when X tends to zero as expected.

3.0 Simplification of PDF of APTLED.

This section presents a simplification of the pdf of APTLED to enable the derivation of some important properties of the distribution. Recall that the pdf of APTLED is given as:

$$f(x)=\frac{\log(\alpha)}{(\alpha-1)}ab^{a}\theta(b+\theta x)^{-(a+1)}\alpha^{\left(1-b^{a}(b+\theta x)^{-a}\right)} \quad (9)$$

For $\alpha>0$ and $\alpha\neq 1$, equation (9) can be expanded using power series as:

$$\alpha^{\left(1-b^{a}(b+\theta x)^{-a}\right)}=\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k}}{k!}\left(1-b^{a}(b+\theta x)^{-a}\right)^{k} \quad (10)$$

Substituting the result in equation (10) into equation (9) and simplifying gives:

$$f(x)=\sum_{k=0}^{\infty}\frac{(\log\alpha)^{k+1}}{(\alpha-1)k!}ab^{a}\theta(b+\theta x)^{-(a+1)}\left(1-b^{a}(b+\theta x)^{-a}\right)^{k} \quad (11)$$

Using the generalized binomial expansion on the last term in equation (11) yields the following:

$$\left(1-b^{a}\left(b+\theta x\right)^{-a}\right)^{k}=\sum_{l=0}^{\infty}(-1)^{l}\binom{k}{l}\left(b^{a}\left(b+\theta x\right)^{-a}\right)^{l} \tag{12}$$

Making use of the result in (12) above in equation (11) and simplifying, we obtain:

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l}\left(\log\alpha\right)^{k+1}a\theta}{\left(\alpha-1\right)k!b^{-a(l+1)}}\left(b+\theta x\right)^{-a(l+1)-1} \tag{13}$$

Hence, equation (13) is the simplified pdf of the alpha power transformed Lomax-exponential distribution (APTLED) which will be used to derive some properties of the APTLED in the next section.

4. Derivation of Properties of Alpha Power Transformed Lomax-exponential Distribution (APTLED)

In this section, some properties of the APTLED are derived and discussed as follows:

4.1 Moments

Let X denote a continuous random variable the n^{th} moment of X is given by;

$$\mu_n^{'}=E\left(X^{n}\right)=\int_{0}^{\infty}x^{n}f\left(x\right)dx \tag{14}$$

Substituting the simplified pdf of the APTLED from equation (13) into equation (14), the n^{th} ordinary moment of the APTLED can be derived as follows:

$$\mu_n^{'}=E\left(X^{n}\right)=\int_{0}^{\infty}x^{n}f\left(x\right)dx=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l}\left(\log\alpha\right)^{k+1}a\theta}{\left(\alpha-1\right)k!b^{-a(l+1)}}\int_{0}^{\infty}x^{n}\left(b+\theta x\right)^{-a(l+1)-1}dx \tag{15}$$

Making use of integration by substitution method in equation (15), we have:

Let $u=b+\theta x\Rightarrow x=-\frac{b}{\theta}\left(1-\frac{u}{b}\right)$ such that $\frac{du}{dx}=\theta\Rightarrow dx=\frac{du}{\theta}$

Substituting for x, u and dx in equation (15) and simplifying; we have:

$$\mu_n^{'}=E\left(X^{n}\right)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l+n}\left(\log\alpha\right)^{k+1}ab^{n-1}}{\left(\alpha-1\right)k!\theta^{n}}\int_{0}^{\infty}\left(\frac{u}{b}\right)^{-a(l+1)-1}\left(1-\frac{u}{b}\right)^{n+1-1}du \tag{16}$$

Recall that $B\left(x,y\right)=\int_{0}^{\infty}t^{x-1}\left(1-t\right)^{y-1}dt=\frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma\left(x+y\right)}$

Using the statement above, the n^{th} ordinary moment of X for the APTLED is obtained as:

$$\mu_n^{'}=E\left(X^{n}\right)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l+n}\left(\log\alpha\right)^{k+1}a\Gamma\left(-a\left(l+1\right)\right)\Gamma\left(n+1\right)}{\left(\alpha-1\right)k!\theta^{n}b^{1-n}\Gamma\left(n-a\left(l+1\right)+1\right)} \tag{17}$$

Using Equation (17), the mean ($\mu_1^{'}$), and variance (σ^2) of the APTLED can be obtained as follows:

$$Mean=E\left(X\right)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l+1}\left(\log\alpha\right)^{k+1}a\Gamma\left(-a\left(l+1\right)\right)}{\left(\alpha-1\right)k!\theta\Gamma\left(2-a\left(l+1\right)\right)} \tag{18}$$

$$V(X)=\sigma^2=E\left(X^2\right)-\left[E(X)\right]^2$$

$$\sigma^2=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l+2}(\log\alpha)^{k+1}2a\Gamma(-a(l+1))}{(\alpha-1)k!\theta^2b^{-2}\Gamma(3-a(l+1))}-\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l+1}(\log\alpha)^{k+1}a\Gamma(-a(l+1))}{(\alpha-1)k!\theta\Gamma(2-a(l+1))}\right)^2$$

(19)

4.2 Moment Generating Function

The moment generating function of a continuous random variable X can be obtained as

$$M_x(t)=E\left[e^{tx}\right]=\int_{-\infty}^{\infty}e^{tx}f(x)dx \tag{20}$$

Using the nth ordinary moment of X in equation (17), the moment generating function of a random variable X for the APTLED can be derived based on power series expansion as follows:

$$M_X(t)=E\left[\mathrm{e}^{tx}\right]=E\left[\sum_{n=0}^{\infty}\frac{(tx)^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^{\infty}x^n f(x)dx=\sum_{n=0}^{\infty}\frac{t^n}{n!}E\left(X^n\right) \tag{21}$$

Substituting the result from equation (21) into equation (20) and simplifying, the moment generating function of the APTLED is obtained as:

$$M_X(t)=E\left[\mathrm{e}^{tx}\right]=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l+n}(\log\alpha)^{k+1}a\Gamma(-a(l+1))\Gamma(n+1)}{(\alpha-1)k!\theta^n b^{1-n}\Gamma(n-a(l+1)+1)}\right] \tag{22}$$

4.3 Characteristics Function

The characteristics function for a continuous random variable X is defined as:

$$\phi_x(t)=E\left(e^{itx}\right)=\int_0^{\infty}e^{itx}f(x)dx \tag{23}$$

The characteristics function for a continuous random variable X can be obtained based on the nth ordinary moment using power series expansion as follows:

$$\phi_X(t)=E\left[\mathrm{e}^{itx}\right]=E\left[\sum_{n=0}^{\infty}\frac{(itx)^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\int_0^{\infty}x^n f(x)dx=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}E\left(X^n\right) \tag{24}$$

Again, substituting for $E\left(X^n\right)$ in equation (24) and simplifying gives the characteristic function of the APTLED as:

$$\phi_X(t)=E\left[\mathrm{e}^{itx}\right]=\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k}{l}\frac{(-1)^{l+n}(\log\alpha)^{k+1}a\Gamma(-a(l+1))\Gamma(n+1)}{(\alpha-1)k!\theta^n b^{1-n}\Gamma(n-a(l+1)+1)}\right] \tag{25}$$

4.4 Reliability analysis of the APTLED.

A derivation and study of the survival function and the hazard rate function is presented in this section. The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x)=1-F(x) \tag{26}$$

Applying the cdf of the APTLED in (26), the survival function for the APTLED is obtained as:

$$S(x)=1-(\alpha-1)^{-1}\left(\alpha^{\left(1-b^{a}(b+\theta x)^{-a}\right)}-1\right) \tag{27}$$

A hazard function is the probability that a component will fail or die for an interval of time. The hazard function is defined as;

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{f(x)}{S(x)} \tag{28}$$

Meanwhile, the expression for the hazard rate of the APTLED is given by:

$$h(x)=\frac{\log(\alpha)ab^{a}\theta(b+\theta x)^{-(a+1)}\alpha^{\left(1-b^{a}(b+\theta x)^{-a}\right)}}{(\alpha-1)\left[(\alpha-1)-\left(\alpha^{\left(1-b^{a}(b+\theta x)^{-a}\right)}-1\right)\right]} \tag{29}$$

The plot of the survival and hazard functions for some chosen parameter values are presented in Figure **2** as shown below:

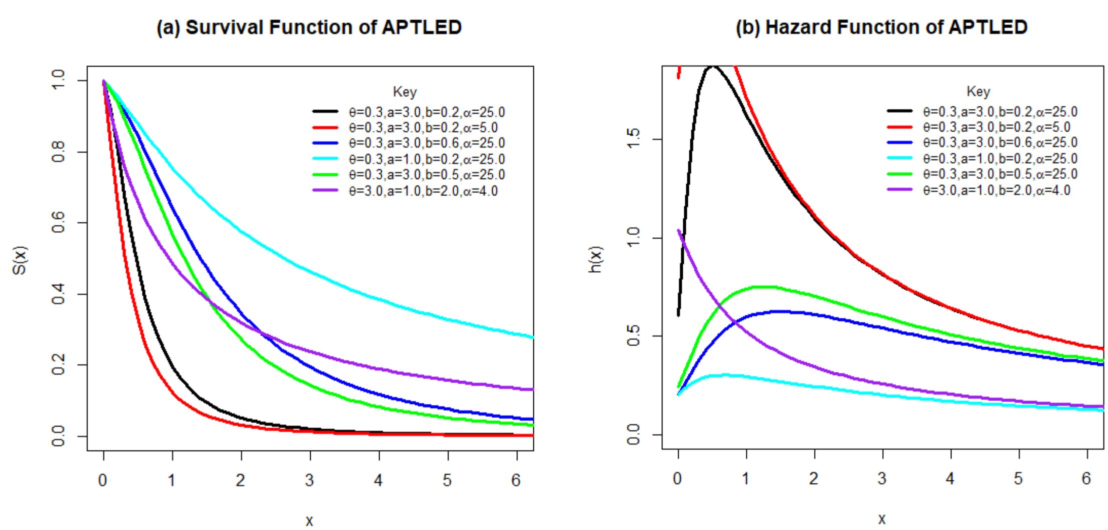

Figure 2: The Survival Function and Hazard Function of APTLED.

The plot in Figure **2(a)** shows that the probability of survival is very high at an initial time or early age and it decreases as time increases up to zero (0) at infinity.

Also, figure **2(b)** revealed that the APTLED has an increasing and decreasing failure rate which implies that the probability of failure for any random variable following APTLED could be increasing and decreasing with time.

5. Maximum Likelihood Estimation of the Parameters of the APTLED

Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the APTLED with unknown parameter, θ, a, b and α defined previously.

The likelihood function of the APTLED using the pdf in equation (8) is given by:

$$L\left(\underline{X} \mid \theta, a, b, \alpha\right) = \left(\frac{\log(\alpha)}{(\alpha-1)} ab^a\theta\right)^n \prod_{i=1}^{n}\left(\left(b+\theta x_i\right)^{-(a+1)} \alpha^{\left(1-b^a\left(b+\theta x_i\right)^{-a}\right)}\right) \qquad (30)$$

Let the natural logarithm of the likelihood function be, $l = \log L\left(\underline{X} \mid \theta, a, b, \alpha\right)$, therefore, taking the natural logarithm of the function in equation (30) above gives:

$$l = n\log\left(\frac{\log\alpha}{\alpha-1}\right) + n\log a + an\log b + n\log\theta - (a+1)\sum_{i=1}^{n}\log\left[b+\theta x_i\right] + \sum_{i=1}^{n}\left[1-b^a\left(b+\theta x_i\right)^{-a}\right]\log\alpha \quad (31)$$

Differentiating l partially with respect to parameters θ, a, b and α gives the following result:

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} - (a+1)\sum_{i=1}^{n}\left[\frac{x_i}{\left(b+\theta x_i\right)}\right] + ab^a\sum_{i=1}^{n}\left[x_i\left(b+\theta x_i\right)^{-a-1}\right]\log(\alpha) \qquad (32)$$

$$\frac{\partial l}{\partial a} = \frac{n}{a} + n\log b - \sum_{i=1}^{n}\log\left[b+\theta x_i\right] + b^a\sum_{i=1}^{n}\left(b+\theta x_i\right)^{-a}\left[\ln\left(b+\theta x_i\right) - \ln b\right]\log(\alpha) \quad (33)$$

$$\frac{\partial l}{\partial b} = \frac{an}{b} - (a+1)\sum_{i=1}^{n}\left[\left(b+\theta x_i\right)^{-1}\right] + ab^a\sum_{i=1}^{n}\left(b+\theta x_i\right)^{-a}\left[\left(b+\theta x_i\right)^{-1} - 1\right]\log(\alpha) \quad (34)$$

$$\frac{\partial l}{\partial\alpha} = \frac{n(\alpha-1)}{\ln\alpha}\left(\frac{\alpha^{-1}(\alpha-1) - \ln\alpha}{(\alpha-1)^2}\right) + \alpha^{-1}\sum_{i=1}^{n}\left[1-b^a\left(b+\theta x_i\right)^{-a}\right] \qquad (35)$$

By equating (32), (33), (34) and (35) to zero and solving for the solution of the non-linear system of equations gives the maximum likelihood estimates of parameters θ, a, b and α respectively.

6. Application to bladder cancer dataset

This section presents an evaluation of the performance of the Alpha power transformed Lomax-exponential distribution (APTLED) compared to other extensions of the exponential and inverse exponential distributions such as the sine Lomax-exponential distribution (SinLomExD) by Joel et al., (2024), transmuted Weibull-exponential distribution (TrWeiExD) by Yahaya and Ieren (2017), the Lomax-exponential distribution (LomExD) by Ieren and Kuhe (2018), transmuted Lindley-exponential distribution (TrLnExD) by Umar *et al*., (2019), the transmuted exponential distribution (TrExD) by Owoloko *et al*. (2015), the odd Lomax-inverse exponential distribution (OLOMINExD) by Ieren et al. (2021), the Lomax distribution (LomD), the inverse exponential distribution (INExD) and exponential distribution (ExD), using a real-life dataset on the emission times of bladder cancer dataset.

The model selection is carried out based upon the value of the log-likelihood function evaluated at the MLEs (ℓ), Akaike Information Criterion, AIC, Consistent Akaike Information Criterion, CAIC, Bayesian Information Criterion, BIC, Hannan Quin Information Criterion, HQIC, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. The details about the statistics A*, W* and K-S are discussed in Chen and Balakrishnan (1995). Meanwhile, the smaller these statistics are, the better the fit of the distribution is.

Dataset: This data represents the remission times (in months) of a random sample of 128 bladder cancer patients. It has previously been used by Lee and Wang (2003), Rady *et al*. (2016), Ieren and

Chukwu (2018), Abdullahi *et al.* (2018) and Ieren and Balogun (2021). It is given and summarized as follows:

0.080, 0.200, 0.400, 0.500, 0.510, 0.810, 0.900, 1.050, 1.190, 1.260, 1.350, 1.400, 1.460, 1.760, 2.020, 2.020, 2.070, 2.090, 2.230, 2.260, 2.460, 2.540, 2.620, 2.640, 2.690, 2.690, 2.750, 2.830, 2.870, 3.020, 3.250, 3.310, 3.360, 3.360, 3.480, 3.520, 3.570, 3.640, 3.700, 3.820, 3.880, 4.180, 4.230, 4.260, 4.330, 4.340, 4.400, 4.500, 4.510, 4.870, 4.980, 5.060, 5.090, 5.170, 5.320, 5.320, 5.340, 5.410, 5.410, 5.490, 5.620, 5.710, 5.850, 6.250, 6.540, 6.760, 6.930, 6.940, 6.970, 7.090, 7.260, 7.280, 7.320, 7.390, 7.590, 7.620, 7.630, 7.660, 7.870, 7.930, 8.260, 8.370, 8.530, 8.650, 8.660, 9.020, 9.220, 9.470, 9.740, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Table 1: Summary Statistics for the bladder cancer dataset

parameters	**n**	**Minimum**	Q_1	**Median**	Q_3	**Mean**	**Maximum**	**Variance**	**Skewness**	**Kurtosis**
Values	128	0.0800	3.348	6.395	11.840	9.366	79.05	110.425	3.3257	19.1537

Based on the descriptive statistics in table 1, it can be seen that the dataset on the remission times of a random sample of 128 bladder cancer patients is heavily skewed to the right or positively skewed which could be good with skewed models.

Table 2: Maximum Likelihood Parameter Estimates based on the bladder cancer data

Distribution	Parameter Estimates			
APTLED	$\hat{\theta}=0.105312590$	$\hat{a}=3.1634829$	$\hat{b}=1.3172828$	$\hat{\alpha}=5.5911918$
SinLomExD	$\hat{\theta}=0.104774074$	$\hat{a}=1.9875491$	$\hat{b}=2.4315592$	-
OLOMINExD	$\hat{\theta}=1.208117086$	$\hat{a}=2.0917350$	$\hat{b}=9.0854609$	-
TrWeiExD	$\hat{\theta}=0.008731942$	$\hat{a}=1.9794500$	$\hat{b}=0.7199023$	$\hat{\lambda}=0.8371726$
TrLnExD	$\hat{\theta}=0.692754881$	$\hat{\alpha}=5.8193159$	$\hat{\lambda}=0.6915196$	-
TrExD	$\hat{\theta}=0.071100252$	$\hat{\lambda}=0.6782716$	-	-
LomExD	$\hat{\theta}=0.167842090$	$\hat{a}=5.6262333$	$\hat{b}=7.6401543$	-
LomD	-	$\hat{a}=1.643146837$	$\hat{b}=9.1200536$	-
INExD	$\hat{\theta}=2.484728884$	-	-	-
ExD	$\hat{\theta}=0.103710244$	-	-	-

Table 3: The statistics ℓ, AIC, CAIC, BIC and HQIC for the bladder cancer data

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*
APTLED	411.6198	831.2396	831.5648	842.6477	835.8747
SinLomExD	414.7788	835.5576	835.7511	844.1137	839.0340
OLOMINExD	425.6260	857.2521	857.4456	865.8082	860.7284
TrWeiExD	479.7421	967.4842	967.8094	978.8923	972.1193
TrLnExD	412.9561	831.9122	832.1057	840.4683	835.3886
TrExD	413.7948	831.5896	831.6856	837.2936	833.9071
LomExD	414.5402	835.0804	835.2739	843.6365	838.5568
LomD	424.4151	852.8301	852.9261	858.5342	855.1477
INExD	413.7948	922.7646	922.7963	925.6166	923.9234
ExD	416.3956	834.7912	834.8229	837.6432	835.9500

Table 4: The A^*, W^*, K-S statistic and P-values for the bladder cancer data

Distribution	A^*	W^*	*K-S*	*P-Value (K-S)*
APTLED	0.1385213	0.02236223	0.08974573	0.2539004
SinLomExD	0.2130714	0.03396388	0.128422	0.02933827
OLOMINExD	1.804141	0.2727428	0.132741	0.02198016
TrWeiExD	0.4703588	0.07547481	0.267779	2.132295e-08
TrLnExD	0.6926032	0.1177998	0.06886163	0.5785225
TrExD	0.3991588	0.06588291	0.1030131	0.132161
LomExD	0.2996184	0.04842946	0.1073644	0.1045662
LomD	0.282648	0.03910122	0.1707996	0.001141952
INExD	6.605157	1.113517	0.231563	2.1849e-06
ExD	0.7081458	0.1179297	0.07963106	0.3914614

The following figure presents a histogram and estimated densities and CDFs of the fitted models to the bladder cancer dataset.

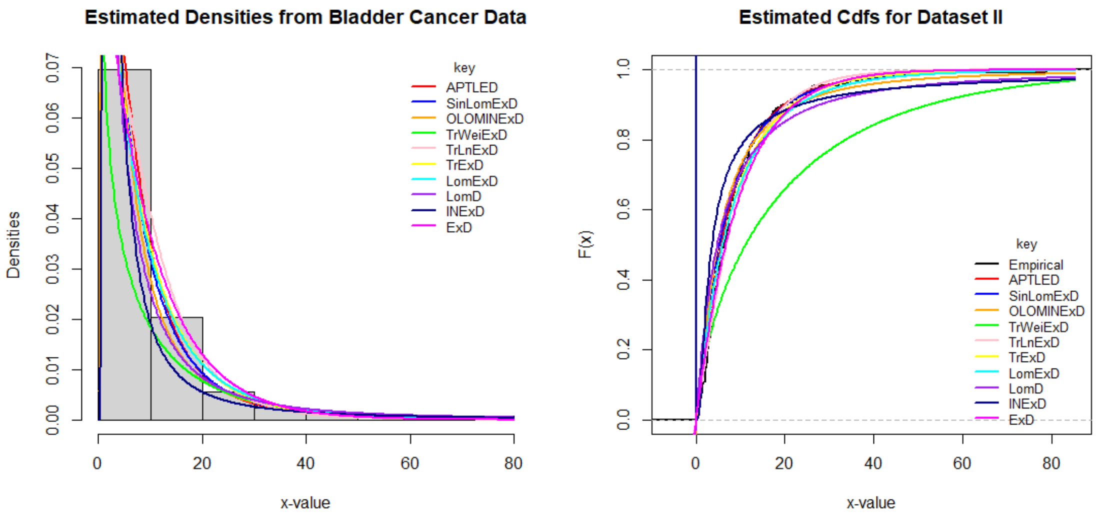

Figure 2: Plots of the estimated densities and CDFs of the fitted distributions to the bladder cancer dataset.

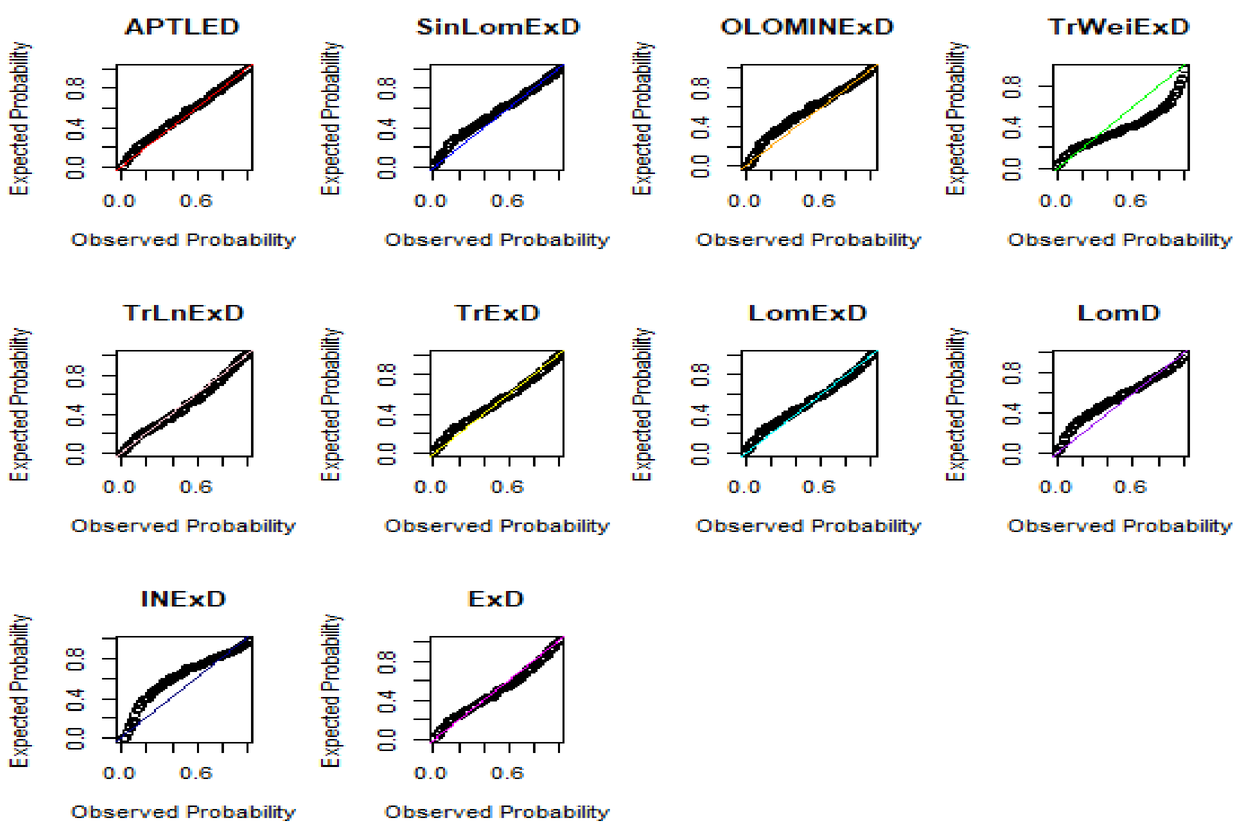

Figure 3: Probability plots for the fitted distributions based on the bladder cancer dataset.

Tables 2 presents the values of the Maximum Likelihood Estimates of the model parameters based on the bladder cancer data, while table 3 presents the values of AIC, CAIC, BIC and HQIC for all the distributions fitted to the data and the values of A*, W* and K-S for the fitted distributions based on the same dataset are provided in Tables 4.

From the results in tables 3 and 4 based on all the model selection measures, it has been found that APTLED has the lowest values of the measures compared to the other fitted distributions and therefore it is considered as the best model to fit the bladder cancer dataset. The results show that the APTLED is better than the other fitted distributions (sine Lomax-exponential distribution (SinLomExD), transmuted Weibull-exponential distribution (TrWeiExD), the Lomax-exponential distribution (LomExD), transmuted Lindley-exponential distribution (TrLnExD), the transmuted exponential distribution (TrExD), the odd Lomax-inverse exponential distribution (OLOMINExD), the Lomax distribution (LomD), the inverse exponential distribution (INExD) and exponential distribution (ExD)). These results are clearly confirmed by the estimated density plots and also the probability plots of the fitted distributions as shown in figures 3 and 4 respectively.

Conclusively, this overall performance of the APTLED is an indication that the alpha power transformed family of distributions is an efficient method for generating continuous distributions as reported in Ceren et al. (2018), Ihtisham et al. (2019), Ahmad et al. (2021), Ahmad et al. (2020), Eghwerido (2021), Mohiuddin and Kannan (2021), Ali et al. (2021), Elbatal et al. (2022), Yakura et al. (2026), as well as Areebah et al. (2026).

7. Summary and Conclusion

This study introduced the Alpha Power Transformed Lomax-Exponential Distribution (APTLED) as a new flexible extension of the Lomax-Exponential distribution through the application of the alpha power transformation technique. The additional shape parameter incorporated into the model significantly increases its flexibility, enabling it to capture a wider range of distributional shapes and hazard rate behaviours than the baseline distribution. Mathematical expressions for the cumulative distribution function, probability density function, moments, moment generating function, characteristic function, survival function and hazard rate function were successfully derived. A simplified representation of the probability density function was also obtained to facilitate the derivation of the distribution's statistical properties, while the maximum likelihood estimation procedure was developed for estimating the unknown model parameters. The practical applicability of the proposed model was demonstrated using remission times of bladder cancer patients and compared with several existing extensions of the exponential and inverse exponential distributions. Based on multiple model selection criteria, including AIC, CAIC, BIC, HQIC, Anderson-Darling, Cramér-von Mises and Kolmogorov-Smirnov statistics, the APTLED consistently outperformed all competing models by producing the smallest goodness-of-fit measures. The graphical assessments through fitted density and probability plots further confirmed its superior performance. These findings demonstrate that the proposed distribution offers greater flexibility and improved modelling accuracy for positively skewed lifetime data. Consequently, the APTLED constitutes a valuable contribution to the family of alpha power transformed distributions and provides researchers and practitioners with an effective model for applications in survival analysis, reliability engineering, medical statistics and other fields involving lifetime data. Future studies may extend the proposed distribution to regression frameworks, Bayesian estimation procedures and censored survival data to further broaden its scope of application.

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