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Probabilists' Hermite Collocation Method For Solving Two Point Higher Order Linear Boundary Problems Of Ordinary Differential Equations

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Abstract

In this paper, a collocation method for approximating higher order boundary value problems of ordinary differential equations (ODEs) utilizing the probabilists' Hermite polynomials of degree 10 as basis function was developed. The scheme was found to have circumvented the Runge phenomenon. Three examples of linear boundary value problems of orders 6,8 and 10 were used as test equations and approximated using the constructed scheme. All the implementations were carried out via MAPLE software and compared with the analytical solutions. The absolute errors shows that the developed method provided good approximation and the results were better than some other numerical solutions available in literature. This demonstrates the reliability and efficiency of the scheme.

Key Words: Collocation method, Probabilists' polynomials, Runge phenomenon

1. Introduction

Higher order boundary value problems (BVPs) arise when mathematical constraints are slightly stretched in order to provide compatibility with physical realities or when the factor that affect the problems are analyzed concurrently (Aydialik and Keris,2006). Higher order BVPs of ODEs have been used to model many physical phenomena. Although there are several applications of such problems, many cannot be solved analytically in terms of elementary functions like polynomials, trigonometric functions, exponential functions, logarithmic functions or their combinations. Even if a differential equation can be solved analytically, considerable effort and sound mathematical theory are often needed and the closed form solution may turn out to be very complicated (*Li et al*., 2018). If the exact solution of a differential equation is too difficult to obtain or not available or take a complicated form, then an approximate solution is preferred.

A variety of approximate methods to solving higher order BVPs abound in literature. These include: the homotopy analysis method (HAM)(Siddiqi and Iftikhar,2013), homotopy perturbation method (HPM)(Razali *et al*.,2016), modified decomposition method (Wazwaz, 2000), Haar Wavelets (Fazal andAli, 2011), second kind Chebychev wavelet (Xu and Zhou, 2015), implicit block method (Adeyeye and Omar, 2019), and the reproducing kernel space method (Akram and Rehman, 2013).

The focus in this research is on the collocation method using orthogonal polynomial as basis functions for approximating higher order BVPs. Classical orthogonal polynomials include Hermite, Jacobi and Laguerre and their special cases include Gegenbuer, Chebychev and Legendre polynomials (Nagaich and Kumar, 2014, Yisa, 2015).

Due to the availability of computers, many researchers in recent years have applied orthogonal polynomials as basis functions for developing approximate methods which are comparable to exact solutions. For example, the Legendre polynomials (Hossain and Islam, 2013), Jacobi polynomials (Aminataei and Imani, 2011, Abd-Elhameed, 2015), Chebychev polynomials (Dolapci, 2014,Yalcinbas *et al*., 2010), Laguerre polynomials(Gurbitz *et al*., 2011) have been utilized as basis functions for developing various collocation methods.

We seek to use the orthogonal Hermite polynomials in developing the collocation methods. There are two ways of standardizing the Hermite polynomials with different variances namely: physicists' Hermite polynomials and the probabilists' (Hermite–Chebychev)Hermite polynomials (Koormwinder *et al*., 2010, Patarroyo, 2020). However, most of the collocation methods in the literature are based on the physicists' Hermite polynomials except for a few as seen in Abada *et al*. (2017), Aboiyar *et al*. (2015) and Luga *et al*. (2019).

In this research, the work done by Luga *et al.*(2019) on the approximation of second order linear BVPs is extended to approximating higher order linear BVPs in ODEs. In their approach, they experimented with various degree of probabilists' Hermite polynomials (degree 3-8.) as basis function and established that, the higher order of the Hermite polynomial of the scheme was more accurate. The numerical results of their collocation method and the ability of the scheme to circumvent the Runge phenomenon (the problem of oscillation associated with the use of higher order polynomial at equally spaced interval) coupled with the successes achieved by those who have implemented it on initial value problems (IVPs) prompted this research. However, we shall employ the probabilists'Hermite polynomial of degree ten as a basis function to construct a collocation method for approximating higher order linear BVPs (order 6,8 and 10). This research has been organized into five sections; section one is the general introduction, section two is the collocation methods, in sections three, results of the developed scheme are presented while sections four and five are discussions of results and conclusion respectively.

2. Methods

2.1 Probabilists' Hermite Polynomials

The Probabilits' Hermite polynomials according to (koormwinder *et al*., 2010) are defined by

$$H_n(x) = (-1)^n exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}\left(exp\left(-\frac{x^2}{2}\right)\right) = \left(x - \frac{d}{dx}\right)^n \cdot \quad (1)$$

the first two members of the polynomial can be generated conveniently from equation (1), while the rest of the members are generated using the recursion formula;

$$H_{n+1}(x) = xH_n - H_n'(x). \quad (2)$$

The first eleven Probabilists' Hermite polynomials generated from equations (1) and (2) are provided below

$$\left.\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= x\\
H_2(x) &= x^2 - 1\\
H_3(x) &= x^3 - 3x\\
H_4(x) &= x^4 - 6x^2 + 3\\
H_5(x) &= x^5 - 10x^3 + 15x\\
H_6(x) &= x^6 - 15x^4 + 45x^2 - 15\\
H_7(x) &= x^7 - 21x^5 + 103x^3 - 105x\\
H_8(x) &= x^8 - 28x^6 + 210x^4 - 420x^2 + 105\\
H_9(x) &= x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x\\
H_{10}(x) &= x^{10} - 45x^8 + 630x^6 - 315x^4 + 4725x^2 - 945
\end{aligned}\right\} \quad (3)$$

2.2 Higher Order Linear Boundary Problems (BVPs)

Let x and $y$denote the independent and dependent variablesrespectively, an nth order linear boundary value problem is given by the formula;

$$y^{(n)}(x) = f\left(x, y(x), y'(x), y''(x) \cdots y^{(n-1)}(x)\right) \quad (4)$$

defined on the interval,$[a, b]$alongside the boundary conditions

$$\left.\begin{array}{llll} y(a) & = A_0\,, & y(b) = B_0 \\ y'(a) & = A_1\,, & y(b) = B_1 \\ y''(a) & = A_2\,, & y(b) = B_2 \\ \vdots & \vdots & \vdots \quad \vdots \\ y^{(n-1)}(a) & = A_{n-1}, & y^{n-1}(b) = B_{n-1} \end{array}\right\} \quad (5)$$

where $A_0,\ A_1,\ A_2, \dots, A_{n-1}$ and $B_0, B_1, B_2 \cdots B_{n-1}$are real finite constants,f is a linear function and n is the order of the differential equation.

In this work, we considered the cases where $n = 6$, 8and 10 together with k given boundary conditions equal to the order of the given ODE.

2.3 Probabilists' Hermite Collocation Method

To develop a collocation method for approximating equation (4) and (5), the first eleven terms of equation (3) are used to form a polynomial of degree 10 as given below:

$$y_n(x) = a_0H_0 + a_1H_1 + a_2H_2 + \cdots + a_{10}H_{10} \quad (6)$$

Equation (6) can be written in compact sigma notation

$$y_n(x) = \sum_{j=0}^{10} a_jH_j(x) \quad (7)$$

2.3.1 Collocation Method for General Linear Higher Order ODEs Using Probalilists Hermite Polynomial

Consider the linear nth order ordinary differential equation of the from:

$$P_n(x)y^{(n)}(x) + P_{n-1}\, y^{(n-1)}(x) + \cdots P_1\,(x)y'(x) + \ P_0(x)y(x) = \ f(x) \quad (8)$$

defined on the internal $a \le x \le b$. $P_n(x)$, $P_{n-1}\,(x) \dots P_0(x)$ are coefficients which may be constants or functions of x. Assumed that (8) can be approximated using;

$$y(x) \approx y_n(x) = \sum_{j=0}^{N} a_jH_j(x) \quad (9)$$

where $H_j, j = 0, 1, \cdots 10$ are as defined in equation (3) and a_j are coefficient which are to be found.

To find $a_{j,}\ j = 0, 1, 2, \ldots, 10,$ equation (9)is differentiated n number of times corresponding to the order of the ODE to be solved and substituted into the left-hand side of the equation (8) which yields

$$P_n(x)\sum_{j=0}^{N} a_j H_j^{(n)}(x) + P_{n-1}(x)\sum_{j=0}^{N} a_j H_j^{(n-1)}(x) + \cdots +$$

$$P_1(x)\sum_{j=0}^{N} a_j H_j'(x) + P_0(x)\sum_{j=0}^{N} a_j H_j(x) = f(x). \quad (10)$$

The like coefficient from equation (11) are collected after which the equation is used for collocation at designated points as shown below

$$\sum_{j=0}^{N} a_j P^*(x) Q_j(x) = f(x) \qquad (11)$$

where $Q_j(x),\ j = 0,1,\ldots n$ are polynomials of different degrees and $P^*(x)$are coefficients which may be constants or functions of x.

2.4 Generating $N + 1$ System of Equations

To get $a_j, j = 0,1 \ldots N, N + 1$ linear system of equations are generated using boundary conditions which are given alongside the differential equation, while the remaining equations are obtained from equation (11) at the collocation points.

2.4.1 Equations generated from the boundary conditions

The following equations are formulated from the boundary conditions:

For the lower boundary conditions, we have,

$$\left.\begin{aligned} y(a) &= \sum_{j=0}^{N} a_j \mathrm{H}(a) = A_0 \\ y'(a) &= \sum_{j=0}^{N} a_j \mathrm{H}'(a) = A_1 \\ y''(a) &= \sum_{j=0}^{N} a_j H_j''(a) = A_2 \\ &\vdots \\ y^{(n-1)}(a) &= \sum_{j=0}^{N} a_j H_j^{(n-1)}(a) = A_{n-1} \end{aligned}\right\} \qquad (12a)$$

Similarly, at the upper boundary, the following equations are generated

$$\left.\begin{aligned} y(b) &= \sum_{j=0}^{N} a_j \mathrm{H}(a) = B_0 \\ y'(b) &= \sum_{j=0}^{N} a_j \mathrm{H}'(a) = B_1 \\ y''(b) &= \sum_{j=0}^{N} a_j H_j''(a) = B_2 \\ &\vdots \\ y^{(n-1)}(b) &= \sum_{j=0}^{N} a_j H_j^{(n-1)}(a) = B_{n-1} \end{aligned}\right\} \qquad (12b)$$

where a and b represent the lower and the upper boundary points respectively, while (12a) and(12b) are the collocation equations using boundary points.

Suppose $k = 2, \dots, n$ are the equations generated from the boundary conditions, the remaining $N + 1 - k$ equations are obtained from the collocation points

$$x_i = \frac{a + (b-a)i}{N-(k-2)}, \qquad i = 1, 2, 3, \dots N-(k-1) \qquad (13)$$

where N is the degree of the approximating polynomial, while k corresponds to the number of boundary conditions. From the cases considered in this work, the order of the differential equation n is equal to the number of boundary conditions.Equation (13) is used to obtaining the collocation points. To this end equation (11) is redefined as

$$\sum_{j=0}^{N} a_j P^*(x) Q_j(x_i) = f(x_i) \qquad (14)$$

In actuality, equation (14) is used to generate the remaining $N + 1 - k$ equations.

If 'k numbers of equations are generated from equations (12a) and (12b) respectively and the remaining $N + 1 - k$ equations are obtained from equation (14), altogether we have $N + 1$ systems of equations to solve in order to get the a_j's, $j = 0,1,2, \dots N$. In general, this can be represented by a square matrix as follows

2.4.2 Equations generated at the collocation points

To get the remaining $N + 1 - k$ equations, equation (11) is collocated at the points,$x = x_i$. These points are obtained from

$$\begin{bmatrix} C_{00} & C_{01} & \cdots & C_{0N} \\ C_{21} & C_{22} & \cdots & C_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ C_{N+11} & C_{N+12} & \cdots & C_{NN} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_N \end{bmatrix} \quad (15)$$

where the coefficient matrix can be generated from;

$$C_{ij=}\begin{cases} H_j^{(i)}(a), & i = 0,1,\dots,\frac{k}{2}-1, \quad j = 0,\dots,N \\ p_j^*(x_i), & i = \frac{k}{2},\dots, \ N-\frac{k}{2}, \quad j = 0\dots,N \\ H_j^{\left(i-\left(N-\frac{k}{2}+1\right)\right)}(b), & i = \left(N-\frac{k}{2}+1\right),\dots,N, \quad j = 0,\dots,N \end{cases} \quad (16)$$

$C_{i,j}$, $d_0, d_1, \dots, d_N$ are constants or functions of the independent variables and $a_0, a_1, \dots, a_N$ are constants to be determined.

Here k is the number of side conditions also called the boundary conditions, corresponding to the order n of the ODE. Substituting, $N = 10$ and $k = n = 6, 8$ and 10 respectively, the various coefficient matrices for higher order ODEs are obtained and used to solve for $a_0, a_1, \dots, a_N$which are substituted into equation (9) to get the approximate solutions for higher order BVPs in ODEs.The approximate solution using a probabilists' Hermite polynomial of degree 10 in form of series is giving by;

$$y^{10}(\underline{x}) = a_0 + a_1x + a_2(x^2-1) + a_3(x^3-3x) + a_5(x^5-10x^3+15x) + a_6 + a_7(x^7-21x^5+105x^3-105x)a_8(x^8-28x^6+210x^4-420x^2+105)+a_9(x^9-36x^7+378x^5-1260x^3+945x)+a_{10}(x^{10}-45x^8+630x^6-3150x^4+4725x^2-945)$$

3. Results

The method developed in the last Section is applied to approximate some problems on 6th , 8th and 10th order linear boundary value problems in ordinary differential equations. The absolute error is used to measure the efficiency of the numerical solutions obtained from the collocation method developed and are further compared with other numerical solutions of methods in literature. These are displayed in Tables (1-3).

Example 1

Consider the 6th order linear boundary value problem solved by Khalid *et al.*(2019)

$$y^{(6)} = y - 4\,exp\,exp\,(x)\,, \quad 0 \le x \le 1$$

Subject to the boundary conditions

$y(0) = 1, y(1) = 0\ y'(0) = 0\ y'(1) = -\ exp\ exp\ (1)\ \ y''(0) = -1\ y''(1) = -2exp(1)\ \}$

This has the following exact solution:

$y(x) = (1 - x)\ exp\ exp\ (x)$.

The comparison of numerical solutions in terms of absolute values are reported in Table 1 for comparison:

Table 1: comparison of numerical Solutions for Example 1

S/N	x_n	Proposed method	$Cubic\ B\ spline\ method$ (Khalid *et al.* 2019)
0	0	1.0020×10^{-17}	Not reported
1	0.1	3.3205×10^{-12}	1.18×10^{-5}
2	0.2	5.6871×10^{-12}	4.29×10^{-5}
3	0.3	3.9195×10^{-12}	8.53×10^{-5}
4	0.4	1.4763×10^{-12}	1.28×10^{-4}
5	0.5	2.1722×10^{-12}	1.59×10^{-4}
6	0.6	5.5411×10^{-12}	1.67×10^{-4}
7	0.7	7.2541×10^{-12}	1.45×10^{-4}
8	0.8	7.9767×10^{-12}	9.47×10^{-5}
9	0.9	4.1402×10^{-12}	3.33×10^{-5}
10	1.0	1.6298×10^{-17}	$not\ reported$

Example 2

Consider the 8th order linear boundary value problem solved by Reddy *et al.* (2017)

$y^{(8)} - y = -8\ exp\ exp\ (x)\ ,\ \ 0 \leq x \leq 1$

Subject to the given boundary conditions

$y(0) = 1, y(1) = 0\ y'(0) = 0\ y'(1) = -\ exp\ exp\ (1)\ \ y''(0) = -1\ y''(1) = -2exp(1)\ y'''(0) = -2\ y'''(0)$

$= -3exp(1)\ \}$

this has the following exact solution;

$y(x) = (1 - x)exp(x).$

The approximate solutions in terms of absolute values are provided in Table 2 for comparison

Table 2: comparison of numerical Solutions for Example 2

S/N	x_n	Proposed Method	$Haar\ \ wavelet$ (Reddy et al, 2017)
0	0	6.2700×10^{-18}.	0
1	0.1	1.4690×10^{-13}	6.3×10^{-11}
2	0.2	3.9871×10^{-12}	6.5×10^{-10}
3	0.3	1.4021×10^{-11}	2.0×10^{-9}
4	0.4	1.1723×10^{-11}	3.3×10^{-9}
5	0.5	9.6118×10^{-12}	3.9×10^{-9}
6	0.6	2.7190×10^{-11}	3.4×10^{-9}
7	0.7	2.1405×10^{-11}	2.0×10^{-11}
8	0.8	5.3188×10^{-11}	6.9×10^{-10}
9	0.9	2.1970×10^{-13}	7.6×10^{-11}
10	1.0	1.0701×10^{-17}	$not\ reported$

Example 3

Consider the 10th order linear boundary value problem presented by Iqbal *et al*. (2015)

$$y^{(10)} = -(x^2 + 19x + 80)\ exp\ exp\ (x)\ ,\ \ 0 \le x \le 1$$

Subject to the given boundary conditions

$$y(0) = 1, y(1) = 0\ y''(0) = 0\ y''(1) = -4e\ y'^{v}(0) = -8\ y'^{v}(1) = -16e\ y^{v\prime}(0) = -24\ y^{v\prime}(1) = -36e\ y^{v'''}(0) = -2\ y^{v'''}(1) = -64e\ \}$$

this has the following exact solution

$y(x) = (1 - x)exp(x)$.

Table 3: Comparison of Numerical Solutions for Example 3

S/N	x_n	Proposed method	$Iqbal\ et\ al$.(2015)	
0	0	8.0000×10^{-21}	$Type\ equation\ here.$	$Type\ equation\ here.$
1	0.1	2.1509×10^{-5}		
2	0.2	4.1015×10^{-5}	2.238×10^{-4}	
3	0.3	5.6671×10^{-5}		
4	0.4	6.6945×10^{-5}	3.745×10^{-4}	
5	0.5	7.0770×10^{-5}		
6	0.6	6.7667×10^{-5}	4.271×10^{-4}	
7	0.7	5.7847×10^{-5}		
8	0.8	4.2185×10^{-5}	3.308×10^{-4}	
9	0.9	2.2232×10^{-5}		
10	1.0	2.1100×10^{-19}		

4. Discussion

Three problems involving sixth, eighth, and tenth order non-homogeneous linear boundary value problem with constant coefficient, together with even number boundary conditions corresponding to the order of the boundary value problems considered. The solution to the boundary value problem was approximated using the proposed scheme. The implementation was carried out in MAPLE. The approximated solutions were in agreement with the analytical results and were compared with other numerical methods in the literature and displayed in Tables 1 - 3. In the case of the sixth order BVP in example 1, the largest absolute error is 7.9767×10^{-12}. This shows a better solutions than that provided by Khalid *et al.* (2019). For Example 2 which is an eighth order boundary value problem, together with eight boundary conditions , the largest absolute error in this case is, $5 \cdot 3188 \times 10^{-11}$, which is relatively better when compared to the numerical solution of the same problem as reported by Reddy *et al*,(2017)which numerical solution has a maximum absolute error of $3 \cdot 9 \times 10^{-9}$. The result in Example 3 which is a tenth order BVP shows that, the largest absolute error observed is, $7 \cdot 0770 \times 10^{-5}$; which shows a relatively accurate result when compared to the absolute error using Polynomial Cubic Spline method (PCSM) as reported by Iqbal *et al*. (2015).

5. CONCLUSION

In this paper, probabilists' Hermite polynomial of degree 10 has been utilized as basis function to construct a collocation method for approximating higher order boundary value problems. The method was observed to be computationally efficient and the algorithm can be implemented on a computer.

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